

SAALSCHÜTZ'S THEOREM AND SUMMATION FORMULAE INVOLVING GENERALIZED HARMONIC NUMBERS

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ABSTRACT. In terms of the derivative operator, integral operator and Saalschütz's theorem, two families of summation formulae involving generalized harmonic numbers are established.

1. INTRODUCTION

For a complex variable x , define the shifted factorial to be

$$(x)_0 = 0 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{with} \quad n \in \mathbb{N}.$$

Following Andrews, Askey and Roy [2, Chapter 2], define the hypergeometric series by

$${}_{{1+r}}F_s \left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ b_1, & \cdots, & b_s \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{(1)_k (b_1)_k \cdots (b_s)_k} z^k,$$

where $\{a_i\}_{i \geq 0}$ and $\{b_j\}_{j \geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then Saalschütz's theorem (cf. [2, p. 69]) can be stated as

$${}_3F_2 \left[\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix} \middle| 1 \right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}. \quad (1)$$

For a complex number x and a positive integer ℓ , define generalized harmonic numbers of ℓ -order to be

$$H_0^{(\ell)}(x) = 0 \quad \text{and} \quad H_n^{(\ell)}(x) = \sum_{k=1}^n \frac{1}{(x+k)^\ell} \quad \text{with} \quad n \in \mathbb{N}.$$

When $x = 0$, they become harmonic numbers of ℓ -order

$$H_0^{(\ell)} = 0 \quad \text{and} \quad H_n^{(\ell)} = \sum_{k=1}^n \frac{1}{k^\ell} \quad \text{with} \quad n \in \mathbb{N}.$$

Fixing $\ell = 1$ in $H_0^{(\ell)}(x)$ and $H_n^{(\ell)}(x)$, we obtain generalized harmonic numbers

$$H_0(x) = 0 \quad \text{and} \quad H_n(x) = \sum_{k=1}^n \frac{1}{x+k} \quad \text{with} \quad n \in \mathbb{N}.$$

2010 Mathematics Subject Classification: Primary 05A10 and Secondary 33C20.

Key words and phrases. Hypergeometric series; Saalschütz's theorem; Derivative operator; Integral operator; Harmonic numbers.

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When $x = 0$, they reduce to classical harmonic numbers

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^n \frac{1}{k} \quad \text{with} \quad n \in \mathbb{N}.$$

For a differentiable function $f(x)$, define the derivative operator \mathcal{D}_x by

$$\mathcal{D}_x f(x) = \frac{d}{dx} f(x).$$

For an integrable function $g(x)$, define the integral operator \mathcal{I}_x by

$$\mathcal{I}_x g(x) = \int_0^x g(x) dx.$$

In order to explain the relation of the derivative operator and generalized harmonic numbers, we introduce the following lemma.

Lemma 1. *Let x and $\{a_j, b_j, c_j, d_j\}_{j=1}^s$ be all complex numbers. Then*

$$\mathcal{D}_x \prod_{j=1}^s \frac{a_j x + b_j}{c_j x + d_j} = \prod_{j=1}^s \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^s \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)}.$$

Proof. It is not difficult to verify the case $s = 1$ of Lemma 1. Suppose that

$$\mathcal{D}_x \prod_{j=1}^m \frac{a_j x + b_j}{c_j x + d_j} = \prod_{j=1}^m \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^m \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)}$$

is true. We can proceed as follows:

$$\begin{aligned} \mathcal{D}_x \prod_{j=1}^{m+1} \frac{a_j x + b_j}{c_j x + d_j} &= \mathcal{D}_x \left\{ \prod_{j=1}^m \frac{a_j x + b_j}{c_j x + d_j} \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \right\} \\ &= \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \mathcal{D}_x \prod_{j=1}^m \frac{a_j x + b_j}{c_j x + d_j} + \prod_{j=1}^m \frac{a_j x + b_j}{c_j x + d_j} \mathcal{D}_x \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \\ &= \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \prod_{j=1}^m \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^m \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)} \\ &\quad + \prod_{j=1}^m \frac{a_j x + b_j}{c_j x + d_j} \frac{a_{m+1} d_{m+1} - b_{m+1} c_{m+1}}{(c_{m+1} x + d_{m+1})^2} \\ &= \prod_{j=1}^{m+1} \frac{a_j x + b_j}{c_j x + d_j} \left\{ \sum_{j=1}^m \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)} + \frac{a_{m+1} d_{m+1} - b_{m+1} c_{m+1}}{(a_{m+1} x + b_{m+1})(c_{m+1} x + d_{m+1})} \right\} \\ &= \prod_{j=1}^{m+1} \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^{m+1} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)}. \end{aligned}$$

This proves Lemma 1 inductively. □

Setting $a_j = 1, b_j = r - j + 1, c_j = 0, d_j = j$ in Lemma 1, it is easy to find that

$$\mathcal{D}_x \binom{x+r}{s} = \binom{x+r}{s} \{H_r(x) - H_{r-s}(x)\},$$

where $r, s \in \mathbb{N}_0$ with $s \leq r$. Besides, we have the following relation:

$$\mathcal{D}_x H_n^{(\ell)}(x) = -\ell H_n^{(\ell+1)}(x).$$

As pointed out by Richard Askey (cf. [1]), expressing harmonic numbers in accordance with differentiation of binomial coefficients can be traced back to Issac Newton. In 2003, Paule and Schneider [7] computed the family of series:

$$W_n(\alpha) = \sum_{k=0}^n \binom{n}{k}^\alpha \{1 + \alpha(n-2k)H_k\}$$

with $\alpha = 1, 2, 3, 4, 5$ by combining this way with Zeilberger's algorithm for definite hypergeometric sums. According to the derivative operator and the hypergeometric form of Andrews' q -series transformation, Krattenthaler and Rivoal [4] deduced general Paule-Schneider type identities with α being a positive integer. More results from differentiation of binomial coefficients can be seen in the papers [9, 13, 14, 15]. For different ways and related harmonic number identities, the reader may refer to [3, 5, 6, 8, 10, 12]. It should be mentioned that Sun [11] showed recently some congruence relations concerning harmonic numbers to us.

Inspired by the work just mentioned, we shall explore, by means of the derivative operator, integral operator and (1), closed expressions for the following two families of series:

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{2x-y+n+k}{k} \binom{y+k}{k}}{(x+k)^2} \frac{\binom{y}{t}}{\binom{y+k}{t}} H_k^{(2)}(x), \\ & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{\binom{y}{t}}{\binom{y+k}{t}} H_k^{(\ell)}(x), \end{aligned}$$

where $t \in \mathbb{N}$. In order to avoid appearance of complicated expressions, our explicit formulae are offered only for $t = 1, 2$ and $\ell = 1, 2, 3, 4$.

2. THE FIRST FAMILY OF SUMMATION FORMULAE INVOLVING GENERALIZED HARMONIC NUMBERS

Theorem 2. *Let x and y be both complex numbers. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{2x-y+n+k}{k} \binom{y+k}{k}}{(x+k)^2} \frac{y}{y+k} H_k^{(2)}(x) \\ &= \frac{\binom{x-y+n}{n}^2}{\binom{x+n}{n}^2} \left\{ H_n^{(2)}(x) - H_n^{(2)}(x-y) \right\}. \end{aligned}$$

Proof. Perform the replacements $a \rightarrow 1+z$, $b \rightarrow y$, $c \rightarrow 1+x$ in (1) to get

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{z+k}{k} \binom{y+k}{k}}{\binom{x+k}{k} \binom{y+z-x-n+k}{k}} \frac{y}{y+k} = \frac{\binom{x-y+n}{n} \binom{x-z-1+n}{n}}{\binom{x+n}{n} \binom{x-y-z-1+n}{n}}. \quad (2)$$

Applying the derivative operator \mathcal{D}_x to both sides of (2), we gain

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{z+k}{k} \binom{y+k}{k}}{\binom{x+k}{k} \binom{y+z-x-n+k}{k}} \frac{y}{y+k} \left\{ H_k(y+z-x-n) - H_k(x) \right\} \\ &= \frac{\binom{x-y+n}{n} \binom{x-z-1+n}{n}}{\binom{x+n}{n} \binom{x-y-z-1+n}{n}} \left\{ H_n(x-y) + H_n(x-z-1) - H_n(x) - H_n(x-y-z-1) \right\}. \end{aligned}$$

The equivalent form of it reads as

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{z+k}{k} \binom{y+k}{k}}{\binom{x+k}{k} \binom{y+z-x-n+k}{k}} \frac{y}{y+k} \sum_{i=1}^k \frac{1}{(x+i)(y+z-x-n+i)} \\ &= \frac{\binom{x-y+n}{n} \binom{x-z-1+n}{n}}{\binom{x+n}{n} \binom{x-y-z-1+n}{n}} \left\{ \frac{H_n(x-y) + H_n(x-z-1)}{2x-y-z+n} - \frac{H_n(x) + H_n(x-y-z-1)}{2x-y-z+n} \right\}. \end{aligned} \quad (3)$$

By means of L'Hôpital rule, we achieve

$$\begin{aligned} & \lim_{z \rightarrow 2x-y+n} \frac{H_n(x-y) + H_n(x-z-1)}{2x-y-z+n} \\ &= \lim_{z \rightarrow 2x-y+n} \frac{H_n^{(2)}(x-z-1)}{-1} \\ &= -H_n^{(2)}(y-x-n-1) \\ &= -H_n^{(2)}(x-y), \end{aligned} \quad (4)$$

$$\begin{aligned} & \lim_{z \rightarrow 2x-y+n} \frac{H_n(x) + H_n(x-y-z-1)}{2x-y-z+n} \\ &= \lim_{z \rightarrow 2x-y+n} \frac{H_n^{(2)}(x-y-z-1)}{-1} \\ &= -H_n^{(2)}(-x-n-1) \\ &= -H_n^{(2)}(x). \end{aligned} \quad (5)$$

Taking the limit $z \rightarrow 2x-y+n$ on both sides of (3) and using (4)-(5), we attain Theorem 2 to complete the proof. \square

Choosing $x = p$, $y = q$ in Theorem 2 with $p, q \in \mathbb{N}_0$ and utilizing (2), we obtain the summation formula involving harmonic numbers of 2-order.

Corollary 3. *Let p and q be both nonnegative integers satisfying $p \geq q$. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{2p-q+n+k}{k} \binom{q+k}{k}}{\binom{p+k}{k}^2} \frac{q}{q+k} H_{p+k}^{(2)} \\ &= \frac{\binom{p-q+n}{n}^2}{\binom{p+n}{n}^2} \left\{ H_{p-q}^{(2)} + H_{p+n}^{(2)} - H_{p-q+n}^{(2)} \right\}. \end{aligned}$$

Theorem 4. *Let x and y be both complex numbers. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{2x-y+n+k}{k} \binom{y+k}{k}}{\binom{x+k}{k}^2} \frac{(y-1)y}{(y+k-1)(y+k)} H_k^{(2)}(x) \\ &= \frac{n^2 + n(1+2x-y) + (1+x-y)^2}{(1+x-y)^2} \frac{\binom{x-y+n}{n}^2}{\binom{x+n}{n}^2} \left\{ H_n^{(2)}(x) - H_n^{(2)}(x-y) \right\} \\ &+ \frac{n^2 + n(1+2x-y)}{(1+x-y)^4} \frac{\binom{x-y+n}{n}^2}{\binom{x+n}{n}^2}. \end{aligned}$$

Proof. Replace c by $1+c$ in (1) to get

$${}_3F_2 \left[\begin{matrix} a, b, -n \\ 1+c, a+b-c-n \end{matrix} \middle| 1 \right] = \frac{(1+c-a)_n (1+c-b)_n}{(1+c)_n (1+c-a-b)_n}.$$

The combination of (1) and the last equation gives

$${}_3F_2 \left[\begin{matrix} a, b, -n \\ 1+c, 1+a+b-c-n \end{matrix} \middle| 1 \right] = \left\{ 1 + \frac{n(c-a-b)}{(c-a)(c-b)} \right\} \frac{(c-a)_n(c-b)_n}{(1+c)_n(c-a-b)_n}. \quad (6)$$

Employ the substitutions $a \rightarrow 1+z$, $b \rightarrow y-1$, $c \rightarrow x$ in (6) to gain

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{z+k}{k} \binom{y+k}{k}}{\binom{x+k}{k} \binom{y+z-x-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} \\ &= \frac{(x-y+1)(x-z-1) + n(x-y-z)}{(x-y+1)(x-z-1+n)} \frac{\binom{x-y+n}{n} \binom{x-z-1+n}{n}}{\binom{x+n}{n} \binom{x-y-z-1+n}{n}}. \end{aligned} \quad (7)$$

Applying the derivative operator \mathcal{D}_x to both sides of (7), we achieve

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{z+k}{k} \binom{y+k}{k}}{\binom{x+k}{k} \binom{y+z-x-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} \left\{ H_k(y+z-x-n) - H_k(x) \right\} \\ &= \frac{(x-y+1)(x-z-1) + n(x-y-z)}{(x-y+1)(x-z-1+n)} \frac{\binom{x-y+n}{n} \binom{x-z-1+n}{n}}{\binom{x+n}{n} \binom{x-y-z-1+n}{n}} \\ & \times \left\{ H_n(x-y) + H_n(x-z-1) - H_n(x) - H_n(x-y-z-1) \right\} \\ &+ \frac{n(z+1)(2x-y-z+n)}{(x-y+1)^2(x-z-1+n)^2} \frac{\binom{x-y+n}{n} \binom{x-z-1+n}{n}}{\binom{x+n}{n} \binom{x-y-z-1+n}{n}}. \end{aligned}$$

The equivalent form of it can be expressed as

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{z+k}{k} \binom{y+k}{k}}{\binom{x+k}{k} \binom{y+z-x-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} \sum_{i=1}^k \frac{1}{(x+i)(y+z-x-n+i)} \\ &= \frac{(x-y+1)(x-z-1) + n(x-y-z)}{(x-y+1)(x-z-1+n)} \frac{\binom{x-y+n}{n} \binom{x-z-1+n}{n}}{\binom{x+n}{n} \binom{x-y-z-1+n}{n}} \\ & \times \left\{ \frac{H_n(x-y) + H_n(x-z-1)}{2x-y-z+n} - \frac{H_n(x) + H_n(x-y-z-1)}{2x-y-z+n} \right\} \\ &+ \frac{n(z+1)}{(x-y+1)^2(x-z-1+n)^2} \frac{\binom{x-y+n}{n} \binom{x-z-1+n}{n}}{\binom{x+n}{n} \binom{x-y-z-1+n}{n}}. \end{aligned}$$

Taking the limit $z \rightarrow 2x-y+n$ on both sides of the last equation and exploiting (4)-(5), we attain Theorem 4 to finish the proof. \square

Selecting $x = p$, $y = q$ in Theorem 4 with $p, q \in \mathbb{N}_0$ and availing (7), we obtain the summation formula involving harmonic numbers of 2-order.

Corollary 5. *Let p and q be both nonnegative integers provided that $p \geq q$. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{2p-q+n+k}{k} \binom{q+k}{k}}{\binom{p+k}{k}^2} \frac{(q-1)q}{(q+k-1)(q+k)} H_{p+k}^{(2)} \\ &= \frac{n^2 + n(1+2p-q) + (1+p-q)^2}{(1+p-q)^2} \frac{\binom{p-q+n}{n}^2}{\binom{p+n}{n}^2} \left\{ H_{p-q}^{(2)} + H_{p+n}^{(2)} - H_{p-q+n}^{(2)} \right\} \\ &+ \frac{n^2 + n(1+2p-q)}{(1+p-q)^4} \frac{\binom{p-q+n}{n}^2}{\binom{p+n}{n}^2}. \end{aligned}$$

Similarly, closed expressions for the following series

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{2x-y+n+k}{k} \binom{y+k}{k}}{\binom{x+k}{k}^2} \frac{\binom{y}{t}}{\binom{y+k}{t}} H_k^{(2)}(x)$$

with $t \geq 2$ can also be derived. The corresponding results will not be displayed here.

3. THE SECOND FAMILY OF SUMMATION FORMULAE INVOLVING GENERALIZED HARMONIC NUMBERS

Theorem 6. *Let x and y be both complex numbers. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{y}{y+k} H_k^{(2)}(x) \\ &= \frac{(-1)^n}{n} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} \left\{ H_n(x-y) - H_n(x) \right\}. \end{aligned}$$

Proof. Perform the replacements $a \rightarrow 1+x$, $b \rightarrow y$, $c \rightarrow 1+z$ in (1) to get

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{x+k}{k} \binom{y+k}{k}}{\binom{z+k}{k} \binom{x+y-z-n+k}{k}} \frac{y}{y+k} = \frac{\binom{z-x-1+n}{n} \binom{z-y+n}{n}}{\binom{z-x-y-1+n}{n} \binom{z+n}{n}}. \quad (8)$$

Applying the derivative operator \mathcal{D}_x to both sides of (8), we have

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{x+k}{k} \binom{y+k}{k}}{\binom{z+k}{k} \binom{x+y-z-n+k}{k}} \frac{y}{y+k} \left\{ H_k(x) - H_k(x+y-z-n) \right\} \\ &= \frac{\binom{z-x-1+n}{n} \binom{z-y+n}{n}}{\binom{z-x-y-1+n}{n} \binom{z+n}{n}} \left\{ H_n(z-x-y-1) - H_n(z-x-1) \right\}. \end{aligned}$$

The equivalent form of it reads as

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{x+k}{k} \binom{y+k}{k}}{\binom{z+k}{k} \binom{x+y-z-n+k}{k}} \frac{y}{y+k} \sum_{i=1}^k \frac{1}{(x+i)(x+y-z-n+i)} \\ &= \frac{\binom{z-x-1+n}{n} \binom{z-y-1+n}{n}}{\binom{z-x-y-1+n}{n} \binom{z+n}{n}} \frac{H_n(z-x-y-1) - H_n(z-x-1)}{y-z}. \end{aligned}$$

Taking the limit $z \rightarrow y-n$ on both sides of the last equation, we gain Theorem 6 to complete the proof. \square

Fixing $x = p$, $y = q$ in Theorem 6 with $p, q \in \mathbb{N}_0$ and using (8), we achieve the summation formula involving harmonic numbers of 2-order.

Corollary 7. *Let p and q be both nonnegative integers satisfying $p \geq q \geq n$. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{q+k}{k}}{\binom{q-n+k}{k}} \frac{q}{q+k} H_{p+k}^{(2)} \\ &= \frac{(-1)^n}{n} \frac{\binom{p-q+n}{n}}{\binom{p+n}{n} \binom{q}{n}} \left\{ H_{p-q+n} - H_{p+n} - H_{p-q} + H_p \right\}. \end{aligned}$$

Theorem 8. *Let x and y be both complex numbers. Then*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{y}{y+k} H_k(x) = \frac{(-1)^n}{n} \frac{1}{\binom{y}{n}} \left\{ 1 - \frac{\binom{x-y+n}{n}}{\binom{x+n}{n}} \right\}.$$

Proof. Applying the integral operator \mathcal{I}_x to both sides of Theorem 6, we attain

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{y}{y+k} \{H_k - H_k(x)\} \\ &= \frac{(-1)^n}{n} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} \Big|_0^x \\ &= \frac{(-1)^n}{n} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} - \frac{(-1)^n}{n} \frac{\binom{-y+n}{n}}{\binom{y}{n}}. \end{aligned} \quad (9)$$

Take the limit $x \rightarrow \infty$ on both sides of (9) to deduce

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{y}{y+k} H_k = \frac{(-1)^n}{n} \frac{1}{\binom{y}{n}} - \frac{(-1)^n}{n} \frac{\binom{-y+n}{n}}{\binom{y}{n}}.$$

The difference of (9) and the last equation creates Theorem 8. \square

Setting $x = p$, $y = q$ in Theorem 8 with $p, q \in \mathbb{N}_0$ and utilizing (8), we obtain the summation formula involving harmonic numbers.

Corollary 9. *Let p and q be both nonnegative integers provided that $q \geq n$. Then*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{q+k}{k}}{\binom{q-n+k}{k}} \frac{q}{q+k} H_{p+k} = \frac{(-1)^n}{n} \frac{1}{\binom{q}{n}} \left\{ 1 - \frac{\binom{p-q+n}{n}}{\binom{p+n}{n}} \right\}.$$

Applying the derivative operator \mathcal{D}_x to both sides of Theorem 8, we get the summation formula involving generalized harmonic numbers of 3-order.

Theorem 10. *Let x and y be both complex numbers. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{y}{y+k} H_k^{(3)}(x) = \frac{(-1)^n}{2n} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} \\ & \times \left\{ [H_n^{(2)}(x-y) - H_n^{(2)}(x)] - [H_n(x-y) - H_n(x)]^2 \right\}. \end{aligned}$$

Choosing $x = p$, $y = q$ in Theorem 10 with $p, q \in \mathbb{N}_0$ and exploiting (8), we gain the summation formula involving harmonic numbers of 3-order.

Corollary 11. *Let p and q be both nonnegative integers satisfying $p \geq q \geq n$. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{q+k}{k}}{\binom{q-n+k}{k}} \frac{q}{q+k} H_{p+k}^{(3)} = \frac{(-1)^n}{2n} \frac{\binom{p-q+n}{n}}{\binom{p+n}{n} \binom{q}{n}} \\ & \times \left\{ [H_{p-q+n}^{(2)} - H_{p+n}^{(2)} - H_{p-q}^{(2)} + H_p^{(2)}] - [H_{p-q+n} - H_{p+n} - H_{p-q} + H_p]^2 \right\}. \end{aligned}$$

Applying the derivative operator \mathcal{D}_x to both sides of Theorem 10, we achieve the summation formula involving generalized harmonic numbers of 4-order.

Theorem 12. *Let x and y be both complex numbers. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{y}{y+k} H_k^{(4)}(x) = \frac{(-1)^n}{6n} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} \\ & \times \left\{ [H_n(x-y) - H_n(x)]^3 + 2[H_n^{(3)}(x-y) - H_n^{(3)}(x)] \right. \\ & \quad \left. - 3[H_n(x-y) - H_n(x)][H_n^{(2)}(x-y) - H_n^{(2)}(x)] \right\}. \end{aligned}$$

Selecting $x = p$, $y = q$ in Theorem 12 with $p, q \in \mathbb{N}_0$ and availing (8), we attain the summation formula involving harmonic numbers of 4-order.

Corollary 13. *Let p and q be both nonnegative integers provided that $p \geq q \geq n$. Then*

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{q+k}{k}}{\binom{q-n+k}{k}} \frac{q}{q+k} H_{p+k}^{(4)} &= \frac{(-1)^n}{6n} \frac{\binom{p-q+n}{n}}{\binom{p+n}{n} \binom{q}{n}} \\ &\times \left\{ [H_{p-q+n} - H_{p+n} - H_{p-q} + H_p]^3 + 2[H_{p-q+n}^{(3)} - H_{p+n}^{(3)} - H_{p-q}^{(3)} + H_p^{(3)}] \right. \\ &\quad \left. - 3[H_{p-q+n} - H_{p+n} - H_{p-q} + H_p][H_{p-q+n}^{(2)} - H_{p+n}^{(2)} - H_{p-q}^{(2)} + H_p^{(2)}] \right\}. \end{aligned}$$

Theorem 14. *Let x and y be both complex numbers. Then*

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} H_k^{(2)}(x) \\ = \frac{(-1)^n (1+x-y+ny)}{n(n-1)(1+x-y)} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} \\ \times \left\{ H_n(x) - H_n(x-y) + \frac{ny}{(1+x-y)(1+x-y+ny)} \right\}. \end{aligned}$$

Proof. Employ the substitutions $a \rightarrow 1+x$, $b \rightarrow y-1$, $c \rightarrow z$ in (6) to obtain

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{x+k}{k} \binom{y+k}{k}}{\binom{z+k}{k} \binom{x+y-z-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} \\ = \frac{(z-x-1)(z-y+1) + n(z-x-y)}{(z-x-1+n)(z-y+1)} \frac{\binom{z-x-1+n}{n} \binom{z-y+n}{n}}{\binom{z-x-y-1+n}{n} \binom{z+n}{n}}. \end{aligned} \quad (10)$$

Applying the derivative operator \mathcal{D}_x to both sides of (10), we get

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{x+k}{k} \binom{y+k}{k}}{\binom{z+k}{k} \binom{x+y-z-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} \left\{ H_k(x) - H_k(x+y-z-n) \right\} \\ = \frac{(z-x-1)(z-y+1) + n(z-x-y)}{(z-x-1+n)(z-y+1)} \frac{\binom{z-x-1+n}{n} \binom{z-y+n}{n}}{\binom{z-x-y-1+n}{n} \binom{z+n}{n}} \\ \times \left\{ H_n(z-x-y-1) - H_n(z-x-1) \right\} \\ - \frac{n(z+n)}{(z-x-1+n)^2(z-y+1)} \frac{\binom{z-x-1+n}{n} \binom{z-y+n}{n}}{\binom{z-x-y-1+n}{n} \binom{z+n}{n}}. \end{aligned}$$

Its equivalent form can be written as

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{x+k}{k} \binom{y+k}{k}}{\binom{z+k}{k} \binom{x+y-z-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} \sum_{i=1}^k \frac{1}{(x+i)(x+y-z-n+i)} \\ = \frac{(z-x-1)(z-y+1) + n(z-x-y)}{(z-x-1+n)(z-y+1)(y-z)} \frac{\binom{z-x-1+n}{n} \binom{z-y-1+n}{n}}{\binom{z-x-y-1+n}{n} \binom{z+n}{n}} \\ \times \left\{ H_n(z-x-y-1) - H_n(z-x-1) \right\} \\ - \frac{n(z+n)}{(z-x-1+n)^2(z-y+1)(y-z)} \frac{\binom{z-x-1+n}{n} \binom{z-y-1+n}{n}}{\binom{z-x-y-1+n}{n} \binom{z+n}{n}}. \end{aligned}$$

Taking the limit $z \rightarrow y-n$ on both sides of the last equation, we gain Theorem 14 to finish the proof. \square

Fixing $x = p$, $y = q$ in Theorem 14 with $p, q \in \mathbb{N}_0$ and using (10), we achieve the summation formula involving harmonic numbers of 2-order.

Corollary 15. *Let p and q be both nonnegative integers satisfying $p \geq q \geq n$. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{q+k}{k}}{\binom{q-n+k}{k}} \frac{(q-1)q}{(q+k-1)(q+k)} H_{p+k}^{(2)} \\ &= \frac{(-1)^n (1+p-q+nq)}{n(n-1)(1+p-q)} \frac{\binom{p-q+n}{n}}{\binom{p+n}{n} \binom{q}{n}} \\ & \times \left\{ H_{p+n} - H_{p-q+n} - H_p + H_{p-q} + \frac{nq}{(1+p-q)(1+p-q+nq)} \right\}. \end{aligned}$$

Theorem 16. *Let x and y be both complex numbers. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} H_k(x) \\ &= \frac{(-1)^n (1+x-y+ny)}{n(n-1)(1+x-y)} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} - \frac{(-1)^n}{n(n-1)} \frac{1}{\binom{y}{n}}. \end{aligned}$$

Proof. Applying the integral operator \mathcal{I}_x to both sides of Theorem 14, we attain

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} \left\{ H_k - H_k(x) \right\} \\ &= \frac{(-1)^{n+1} (1+x-y+ny)}{n(n-1)(1+x-y)} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} \Big|_0^x \\ &= \frac{(-1)^n (1-y+ny)}{n(n-1)(1-y)} \frac{\binom{-y+n}{n}}{\binom{y}{n}} - \frac{(-1)^n (1+x-y+ny)}{n(n-1)(1+x-y)} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}}. \end{aligned} \quad (11)$$

Take the limit $x \rightarrow \infty$ on both sides of (11) to derive

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} H_k \\ &= \frac{(-1)^n (1-y+ny)}{n(n-1)(1-y)} \frac{\binom{-y+n}{n}}{\binom{y}{n}} - \frac{(-1)^n}{n(n-1)} \frac{1}{\binom{y}{n}}. \end{aligned}$$

The difference of (11) and the last equation produces Theorem 16. \square

Setting $x = p$, $y = q$ in Theorem 16 with $p, q \in \mathbb{N}_0$ and utilizing (10), we obtain the summation formula involving harmonic numbers.

Corollary 17. *Let p and q be both nonnegative integers provided that $q \geq n$. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{q+k}{k}}{\binom{q-n+k}{k}} \frac{(q-1)q}{(q+k-1)(q+k)} H_{p+k} \\ &= \frac{(-1)^n (1+p-q+nq)}{n(n-1)(1+p-q)} \frac{\binom{p-q+n}{n}}{\binom{p+n}{n} \binom{q}{n}} - \frac{(-1)^n}{n(n-1)} \frac{1}{\binom{q}{n}}. \end{aligned}$$

Applying the derivative operator \mathcal{D}_x to both sides of Theorem 14, we get the summation formula involving generalized harmonic numbers of 3-order.

Theorem 18. *Let x and y be both complex numbers. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} H_k^{(3)}(x) \\ &= \frac{(-1)^n (1+x-y+ny)}{2n(n-1)(1+x-y)} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} \{A_n(x, y) + B_n(x, y)\}, \end{aligned}$$

where the two symbols on the right hand side stand for

$$\begin{aligned} A_n(x, y) &= [H_n^{(2)}(x) - H_n^{(2)}(x-y)] + \frac{2ny}{(1+x-y)^2(1+x-y+ny)}, \\ B_n(x, y) &= [H_n(x) - H_n(x-y)] \left[H_n(x) - H_n(x-y) + \frac{2ny}{(1+x-y)(1+x-y+ny)} \right]. \end{aligned}$$

Choosing $x = p$, $y = q$ in Theorem 18 with $p, q \in \mathbb{N}_0$ and exploiting (10), we gain the summation formula involving harmonic numbers of 3-order.

Corollary 19. *Let p and q be both nonnegative integers satisfying $p \geq q \geq n$. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{q+k}{k}}{\binom{q-n+k}{k}} \frac{(q-1)q}{(q+k-1)(q+k)} H_{p+k}^{(3)} \\ &= \frac{(-1)^n (1+p-q+nq)}{2n(n-1)(1+p-q)} \frac{\binom{p-q+n}{n}}{\binom{p+n}{n} \binom{q}{n}} \{C_n(x, y) + D_n(x, y)\}, \end{aligned}$$

where the corresponding expressions are

$$\begin{aligned} C_n(p, q) &= [H_{p+n}^{(2)} - H_{p-q+n}^{(2)} - H_p^{(2)} + H_{p-q}^{(2)}] + \frac{2nq}{(1+p-q)^2(1+p-q+nq)}, \\ D_n(p, q) &= [H_{p+n} - H_{p-q+n} - H_p + H_{p-q}] \\ &\quad \times \left[H_{p+n} - H_{p-q+n} - H_p + H_{p-q} + \frac{2nq}{(1+p-q)(1+p-q+nq)} \right]. \end{aligned}$$

Applying the derivative operator \mathcal{D}_x to both sides of Theorem 18, we achieve the summation formula involving generalized harmonic numbers of 4-order.

Theorem 20. *Let x and y be both complex numbers. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{(y-1)y}{(y+k-1)(y+k)} H_k^{(4)}(x) \\ &= \frac{(-1)^n}{6n(n-1)(1+x-y)} \frac{\binom{x-y+n}{n}}{\binom{x+n}{n} \binom{y}{n}} \\ &\quad \times \left\{ (1+x-y+ny)E_n(x, y) + \frac{3ny}{1+x-y}F_n(x, y) + G_n(x, y) \right\}, \end{aligned}$$

where the three symbols on the right hand side stand for

$$\begin{aligned} E_n(x, y) &= [H_n(x) - H_n(x-y)]^3 + 2[H_n^{(3)}(x) - H_n^{(3)}(x-y)] \\ &\quad + 3[H_n(x) - H_n(x-y)][H_n^{(2)}(x) - H_n^{(2)}(x-y)], \\ F_n(x, y) &= [H_n(x) - H_n(x-y)]^2 + [H_n^{(2)}(x) - H_n^{(2)}(x-y)], \\ G_n(x, y) &= \frac{6ny}{(1+x-y)^2} [H_n(x) - H_n(x-y)] + \frac{6ny}{(1+x-y)^3}. \end{aligned}$$

Selecting $x = p$, $y = q$ in Theorem 20 with $p, q \in \mathbb{N}_0$ and availing (10), we attain the summation formula involving harmonic numbers of 4-order.

Corollary 21. *Let p and q be both nonnegative integers provided that $p \geq q \geq n$. Then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{q+k}{k}}{\binom{q-n+k}{k}} \frac{(q-1)q}{(q+k-1)(q+k)} H_{p+k}^{(4)} \\ &= \frac{(-1)^n}{6n(n-1)(1+p-q)} \frac{\binom{p-q+n}{n}}{\binom{p+n}{n} \binom{q}{n}} \\ & \times \left\{ (1+p-q+nq)U_n(p, q) + \frac{3nq}{1+p-q}V_n(p, q) + W_n(p, q) \right\}, \end{aligned}$$

where the corresponding expressions are

$$\begin{aligned} U_n(p, q) &= [H_{p+n} - H_{p-q+n} - H_p + H_{p-q}]^3 + 2[H_{p+n}^{(3)} - H_{p-q+n}^{(3)} - H_p^{(3)} + H_{p-q}^{(3)}] \\ &+ 3[H_{p+n} - H_{p-q+n} - H_p + H_{p-q}][H_{p+n}^{(2)} - H_{p-q+n}^{(2)} - H_p^{(2)} + H_{p-q}^{(2)}], \\ V_n(p, q) &= [H_{p+n} - H_{p-q+n} - H_p + H_{p-q}]^2 + [H_{p+n}^{(2)} - H_{p-q+n}^{(2)} - H_p^{(2)} + H_{p-q}^{(2)}], \\ W_n(p, q) &= \frac{6nq}{(1+p-q)^2} [H_{p+n} - H_{p-q+n} - H_p + H_{p-q}] + \frac{6nq}{(1+p-q)^3}. \end{aligned}$$

Closed expressions for the following series

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{y+k}{k}}{\binom{y-n+k}{k}} \frac{\binom{y}{t}}{\binom{y+k}{t}} H_k^{(\ell)}(x)$$

with $t \geq 2$ and $\ell \geq 5$ can also be given in the same way. The corresponding conclusions will not be laid out in the paper.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (No. 11301120).

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